Quantitative Analysis for the Quality of Ad Hoc Networks

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Abstract

This paper investigates methods to measure the connection-level quality of mobile ad hoc networks. Using an exactly solvable 1-dimensional model, we derive a set of closed formulas that describe the quality of MANETs concerning connectivity, stability, and coverage. These results allow us to predict the minimum number of network nodes for a certain quality level (the "critical mass") without the need for numerical simulations.

I. Introduction

Mobile ad hoc networks are self-organizing structures in which mobile nodes are temporarily connected without the aid of any fixed infrastructure or centralized administration. Mobile ad hoc networks promise a high potential for mobile and ubiquitous computing scenarios. As mobile devices and wireless networks get increasingly powerful, many researchers expect that mobile ad hoc networks (in the following called *MANETs*) will play an important role for mobile users in the future. Many encouraging simulations affirm this view.

Even though MANETs are not planned like traditional networks, reasonable MANETs have to fulfill various conditions to be useful. Nodes, e.g., have to use the same radio technology and the same ad hoc routing protocol. Inside the covered area, a sufficient number of nodes (which we call the *critical mass*) is required to get a useful connectivity. In this paper, we abstract from specific routing protocols, network hardware, etc. and describe the quality of a network with the help of a set of measures that consider the number of nodes, the communication range and the geometric size of the network's area. For each measure, a closed formula is provided and verified with the help of simulation results. As an important result, the expected quality of a MANET can be evaluated with the help of our formulas without the need to install a MANET in the field or even simulate the MANET inside a network simulator.

In order to derive explicit results for the quality levels, we make use of a 1-dimensional MANET model that was recently shown to be exactly solvable [4]; here, network nodes are deployed at random along a straight line. Disregarding boundary effects, this model allows the direct description of quantities that are related to the next-neighbor distances of nodes, which makes it particularly suited for a discussion of node-to-node connectivity.

The paper is organized as follows: In Sec. III, we introduce a general notion of quality measures for ad hoc networks and give some examples of such measures. They will be analyzed in detail in Sec. IV, where we will obtain analytical results in the 1-dimensional model, which will then be compared to simulation results in Sec. V. In Sec. VI, we apply our results in order to give quantitative predictions for the critical mass of 1-dimensional MANETs. Applications to MANETs in 2 and 3 dimensions are discussed in Sec. VII. We end with a brief outlook in Sec. VIII.

This paper is partially based on a thesis by one of the authors [4].

II. Related Work

In the literature, the quality of MANETs is usually described by the probability that the network is connected, i.e. that there is a multi-hop network path between each pair of nodes. Some early works [11, 12] established asymptotic estimates for the probability of connectedness in 1-dimensional systems and conjectured analogous results for 2-dimensional systems. The nodes were distributed on an area (or line segment) according to a Poisson process of homogeneous density. The authors also dealt with the probability that the area is completely covered by the MANET, i.e. that each point is in the range of at least one network node. Recently, the results for 2-dimensional systems were made precise by Xue and Kumar [15]. Denoting the number of network nodes by n, they show that the number of neighbors each node connects to must grow by a factor $\Theta(\ln n)$ in the limit $n \to \infty$

if one wants to keep connectedness. Similar results were found when the radio range is varied as $n \to \infty$ [6, 9].

The probability of connectedness was also considered by Santi and Blough [14]. The authors derive asymptotic estimates mainly for the 1-dimensional system and present numerical (simulation) results also for 2- and 3-dimensional systems. Bettstetter [2] introduced the more general condition of k-connectedness. (The network is called *k-connected* if there are at least k independent network paths between each node pair.) Using results from the theory of random graphs [10], he established analytical estimates for the 2-dimensional case and verified them with numerical results. He also calculated the probability that none of the nodes is completely isolated in the network.

More general quality measures have been defined by one of the authors [13] and investigated in a numerical simulation. Here, alternative quantities based on the number of separated network segments, the size of these segments, and their dependence on changes in the network (e.g. a node being switched off), are taken as a quality indicator.

Further, an analytical estimate of the bandwidth available to each node is given by Gupta and Kumar [7]. The authors show that this bandwidth is of the order $W/\sqrt{n \ln n}$, where W is the bandwidth of the point-to-point links; thus, the throughput that each node is able to use rapidly decreases with the network size.

III. Measuring MANET's properties

III.A. The general setting

In the following, we consider a MANET with a fixed number n of network nodes which are deployed in an s-dimensional region (i.e., for $s = 1$, along a line or curve, for $s = 2$, within an area, and for $s = 3$, in a volume). The positions of the network nodes usually cannot be controlled, and so we assume them to be random according to some probability distribution. All quantities Q describing the state of the network (we call them *quality measures* for our purposes) thus become random variables, and we are usually interested in their expectation value $\bar{Q} = \mathrm{E}[Q]$, i.e., the mean quality of the network.

Our analysis restricts to the situation of the network at fixed time. This may seem to be a bit contrary to our goal to describe mobile ad hoc networks. However, mobility of the nodes does not imply that the probability to find a node within a specific region varies with time. Further, we abstract from specific routing protocols and technical details, and merely focus on the question whether network nodes are connected to each other. Neglecting the aspects of link throughput or packet loss rates, it is justified to assume that all nodes have an equal circular radio range r, below which two nodes are connected to each other, and beyond which the communication breaks down immediately. We will now present some specific quality measures based on this model.

III.B. MANET's properties

What type of quantity should be used to measure the quality of ad hoc networks? One natural choice for such a quality measure is the probability that the network is connected. However, one may go further and ask the following questions:

- *When a new node enters the area of a MANET, how high is the probability to be instantly connected?*
- *Once a node is connected to the MANET, how many nodes can it access, or in turn, how many nodes can access the new node?*
- *Once a node accessed another node, how stable is the communication link?*

In order to answer these questions, the following quality measures have been introduced in [13].

Area Coverage. *Area coverage* is the area A_{covered} covered by the range of at least one MANET node (Fig. 1), divided by the total area A_{total} of the system:

$$Q_{\text{Coverage}} := \frac{A_{\text{covered}}}{A_{\text{total}}}. \qquad (1)$$

Its expectation value may be understood as the probability that an external network node, with its position randomly chosen, will be able to connect to at least one of the n nodes of the MANET.

Segmentation. The *segmentation* of a MANET counts the number of disconnected segments in the network, i.e. the number of subgraphs into which the network graph is separated: We set

$$Q_{\text{Segmentation}} := \frac{\text{\# of network segments}}{n}. \qquad (2)$$

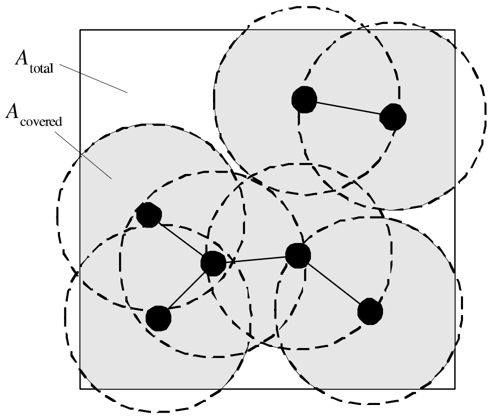


Figure 1: The MANET's covered and total area

Vulnerability. The next quality measure we consider is related to the question how much the network quality or topology changes when a single node is removed from the network. We define the *importance* of the network node with number j as

$$I_j := \max\{0, (\text{\# segments with node } j \text{ removed}) - (\text{\# segments})\}; \quad (3)$$

i.e. I_j is the number of network segments which are created by switching off node j in the current configuration. Nodes with $I_j > 0$ make the network "vulnerable" against changes (Fig. 2).

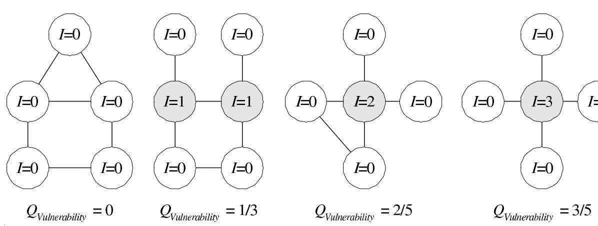


Figure 2: The importance of nodes and the MANET's vulnerability

This motivates to define the *vulnerability* of the network as

$$Q_{\text{Vulnerability}} := \frac{1}{n} \sum_{j=1}^{n} I_j. \quad (4)$$

Reachability. The *reachability* is concerned with the number of nodes that can be reached from a given node (in a multi-hop fashion), or, alternatively speaking, with the size of the segments of the network. We define the reachability of some fixed node j as

$$R_j := \frac{\text{\# of nodes reachable from node } j}{n}. \quad (5)$$

Here we do not count the node itself as reachable. Our intended quality measure, the average reachability, is then defined as follows.

$$Q_{\text{Reachability}} := \frac{1}{n} \sum_{j=1}^{n} R_j. \quad (6)$$

As said before, the above four quality measures have already been considered in [13]; but only simulation results were obtained for them in specific models.

III.C. Scalability

While the choice of specific quality measures naturally is very dependent on the usage scenario and application, there is one property that we wish to discuss in a general context: It is related to the scaling behavior of the system, since we are usually interested in the limit of large MANETs ($n \to \infty$).

A MANET in our sense is characterized by its node number n, the radio range r, and a characteristic length ℓ of the deployment region (say, its diameter); we neglect other possible parameters in this section. A quality measure Q also depends on these parameters; we denote this dependency as $Q^{(n,r,\ell)}$. Varying the system parameters n, r, ℓ, the size of the deployment region scales as ℓ^s, while the area covered by one MANET node scales as r^s. Thus, the quantity

$$\nu = nr^s/\ell^s \quad (7)$$

can be interpreted (up to a constant) as the mean number of nodes reachable from some point within the deployment region.

We are particularly interested in quality measures Q which are "intrinsic" properties of the system, i.e. related to its behavior in the bulk. Such a quality level should not change, if we divide the system into equal parts, or join several identical systems into one. Since it is precisely the quantity ν which stays invariant under such operations, one may expect that $\bar{Q}$ would depend on ν only – rather than on n, r, and ℓ individually –, at least in the limit of large systems.

Let us express this more formally. We describe the limit of large systems as $n \to \infty$ and consider $r = r_n$ and $\ell = \ell_n$, and hence also $\nu_n = nr_n^s/\ell_n^s$, as functions of n. Motivated by the considerations above, we say that a quality measure Q is *intensive*, if it shows the behavior

$$\mathrm{E}[Q^{(n,r_n,\ell_n)}] \to \tilde{Q}(\nu) \quad \text{as } n \to \infty, \nu_n \to \nu, \quad (8)$$

where the limit function $\tilde{Q}$ is nontrivial, i.e., not constant for all limit values ν.

By this definition, we do not intend to say that only intensive quality measures are relevant in practice, or that non-intensive measures are not meaningful. In fact, such non-intensive quality measures may be required for some applications. However, one should keep in mind that they may not scale well for large systems: For example, if we need $\nu_n \to \infty$ in order to keep the quality level of the system constant as $n \to \infty$, then this means that the average number of nodes that are reachable from some point needs to grow arbitrarily in the limit; thus we are likely to run out of local channel capacity. Hence applications which rely on a high quality level with respect to non-intensive measures may not be feasible in networks with a high node number.

IV. An Analytical Approach

We now proceed to a specific MANET model which we analyze in detail. This model restricts to the case $s = 1$, i.e., the nodes are located along a straight line. One might think here of pedestrians moving along sidewalks, or of cars on a road, that carry wireless devices. We consider a fixed number n of network nodes that are deployed at random to an interval $[0, \ell]$, where the locations of individual nodes are independent of each other, and no point within the interval is preferred. In this simple situation, we can derive explicit analytical results for our intended quality measures.

Technically, we use the same model as exposed in [4]: The locations of nodes are described by the sample space

$$\Omega_n = [0, \ell]^n, \tag{9}$$

considered with the equal distribution. Then quality measures are functions $Q^{(n,r,\ell)} : \Omega_n \to \mathbb{R}$. We shall briefly recall the terminology and results of [4, Chap. 3], since our methods are based on these techniques.

First, it is useful to note that all of our quality measures Q fulfill the property

$$\forall \lambda > 0 : \; Q^{(n,r,\ell)}(\boldsymbol{x}) = Q^{(n,\lambda r,\lambda \ell)}(\lambda \boldsymbol{x}). \tag{10}$$

[To see this, note that $Q^{(n,\lambda r,\lambda \ell)}(\lambda \boldsymbol{x})$ describes a situation where all coordinates, lengths, etc. of the system have been scaled by a factor of λ. By geometric similarity, this transformation does not change the number of network segments, nor the fact whether two nodes are connected.] Then a simple scaling argument shows us that the expectation value $\mathrm{E}[Q^{(n,r,\ell)}]$ depends on n and the "normalized radio range" $\rho = r/\ell$ only; hence we may simply assume a radio range of ρ and a sample space of $\Omega_n = [0, 1]^n$.

Further simplification is possible, if we eliminate boundary effects by introducing *periodic boundary conditions:* we identify the left and right end points of the region $[0, 1]$ with each other, allowing nodes to connect "via the boundary" (cf. Fig. 3).[1] More formally, this means that two nodes i and j are connected not only if $|x_i - x_j| < \rho$, but also if $|x_i - x_j \pm 1| < \rho$. This introduces an additional translation symmetry which we can exploit to simplify the situation.

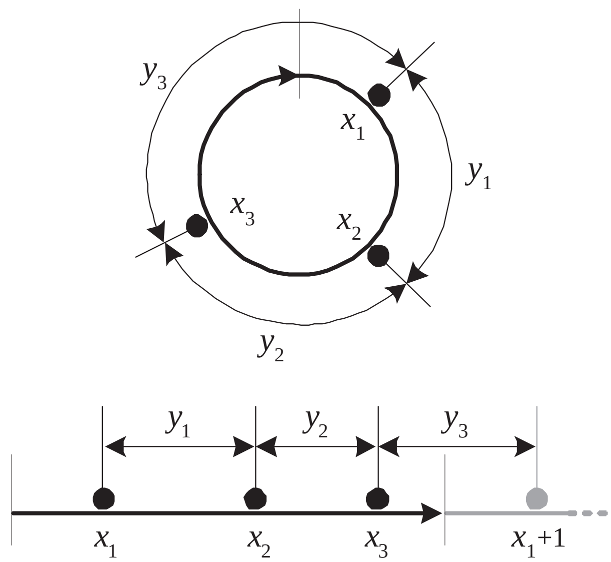


Figure 3: Periodic boundary for three nodes

Since our quality measures do not change when permuting the node coordinates, we can always assume that the components of $\boldsymbol{x} \in \Omega_n$ are written in sorted order, i.e. $x_1 \leq \ldots \leq x_n$. We use this ordering to introduce the next-neighbor distances

$$\begin{aligned} y_i &= x_{i+1} - x_i \quad \text{for } i \in \{1, \ldots, n-1\}, \\ y_n &= x_1 - x_n + 1. \end{aligned} \tag{11}$$

It was shown in [4] that these next-neighbor distances are distributed equally over the top surface of the n-dimensional standard simplex, i.e. over the set

$$T_n = \big\{ \boldsymbol{y} \in [0, 1]^n \;\big|\; \sum_{i=1}^{n} y_i = 1 \big\}. \tag{12}$$

If Q now is a quality measure that can be expressed in terms of the next-neighbor coordinates, i.e. as a function of $\boldsymbol{y}$, then its expectation value can be calculated as

$$\mathrm{E}[Q] = \int d\mu_n^{T-\mathrm{eq}}(\boldsymbol{y}) \; Q(\boldsymbol{y}), \tag{13}$$

[1] Note that this is not the same as counting the node indices $\bmod\ n$, even if we will make use of similar concepts later on.

where $\mu_n^{T-\mathrm{eq}}$ is the probability measure of equal distribution on T_n.

For our quality measures related to node-to-node connectivity, the function $Q(\boldsymbol{y})$ usually involves terms of the type $\theta(\rho - y_j)$, where θ is the Heaviside theta function,

$$\theta(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases} \tag{14}$$

Integrals of functions of this kind over T_n can often be explicitly solved; cf. also the Appendix.

The above formalism allows us to obtain explicit results for quantities related to the next-neighbor distances. Let us recall the main result of [4]: If k-DISCONN is the event that the network is disconnected at exactly k places, then its probability is

$$P_{\text{k-DISCONN}} = \sum_{j=k}^{[1/\rho]} (-1)^{j-k} \binom{j}{k} \binom{n}{j} (1 - j\rho)^{n-1}. \tag{15}$$

Here $[1/\rho]$ denotes the greatest integer less or equal to $1/\rho$. In the limit $n \to \infty$ and $n\rho - \ln n \to \eta$, one has

$$P_{\text{k-DISCONN}}^{(n,\rho)} \xrightarrow{n\to\infty} \frac{e^{-\eta k}}{k!} e^{-(e^{-\eta})}. \tag{16}$$

IV.A. Connectedness

As a first quality measure, let us consider the probability that the network is connected. Accounting for the periodic boundary conditions, we consider the network as connected, if each node is connected to both of its nearest neighbors. More formally, we define the quality measure $Q_{\text{Connectedness}}$ to be the characteristic function of the event 0-DISCONN above. Using Eq. (15), we directly see that

$$\mathrm{E}[Q_{\text{Connectedness}}] = \sum_{j=0}^{[1/\rho]} (-1)^j \binom{n}{j} (1 - j\rho)^{n-1}. \tag{17}$$

In the limit $n \to \infty$ and $n\rho - \ln n \to \eta$, one obtains from Eq. (16):

$$\mathrm{E}[Q_{\text{Connectedness}}] \xrightarrow{n\to\infty} e^{-(e^{-\eta})}. \tag{18}$$

The right-hand side approaches 0 if we choose large negative values for η. A limit $n \to \infty$, $n\rho \to$ const will involve even smaller values for $\rho(n)$; and since $Q_{\text{Connectedness}}^{(n,\rho)}$ is monotone in ρ at fixed n, it follows that $\mathrm{E}[Q_{\text{Connectedness}}] \to 0$ whenever $n\rho \to \nu$. Thus, connectedness is not an intensive quality measure, and applications relying on connectedness of the network may reach scalability bounds.

IV.B. Area Coverage

Area coverage can in fact be calculated without referring to next-neighbor distances: We can certainly say that $\mathrm{E}[1 - Q_{\text{Coverage}}]$ is the probability that a dedicated point, distributed at random to $[0, \ell]$, will fall into the range of *none* of the MANET nodes. The probability for the dedicated point to fall into the range of one specific node, however, is simply 2ρ, where $\rho = r/\ell$ as usual, and where we assume $\rho < \frac{1}{2}$. Due to the independent distribution of network nodes, we obtain

$$\mathrm{E}[Q_{\text{Coverage}}] = 1 - (1 - 2\rho)^n. \tag{19}$$

Using the Taylor approximation

$$\ln(1 - 2\rho)^n = -2n\rho + O(n\rho^2), \tag{20}$$

one obtains in the limit $n \to \infty$ and $n\rho \to \nu > 0$:

$$\mathrm{E}[Q_{\text{Coverage}}] \to 1 - e^{-2\nu}. \tag{21}$$

Thus, area coverage is an intensive quality measure.

IV.C. Segmentation

Within our 1-dimensional system, it is easy to derive an explicit expression for the segmentation: We know that the event k-DISCONN corresponds to a situation with exactly k network segments, where we count the connected situation as 0 segments in order to account for the periodic boundary conditions. Since these events are disjoint, and since their union (over $k = 0 \ldots n$) exhausts the sample space Ω_n, it follows that

$$Q_{\text{Segmentation}} = \frac{1}{n} \sum_{k=0}^{n} k \chi_{\text{k-DISCONN}}. \tag{22}$$

Here $\chi_{\text{k-DISCONN}}$ denotes the characteristic function of the event k-DISCONN; its expectation value $P_{\text{k-DISCONN}}$ is known from Eq. (15). Thus, the expectation value of the segmentation is

$$\mathrm{E}[Q_{\text{Segmentation}}] = \frac{1}{n} \sum_{k=0}^{n} \sum_{j=k}^{[1/\rho]} (-1)^{j-k} k \binom{j}{k} \binom{n}{j} (1 - j\rho)^{n-1}. \tag{23}$$

In the sum over j, we may replace the lower limit with 0, since the binomial coefficient $\binom{j}{k}$ vanishes for $j < k$. We may then exchange the order of summation and obtain

$$\mathrm{E}[Q_{\text{Segmentation}}] = \frac{1}{n} \sum_{j=0}^{[1/\rho]} (-1)^j \binom{n}{j} (1 - j\rho)^{n-1} \times \sum_{k=0}^{n} (-1)^k k \binom{j}{k}. \tag{24}$$

Likewise, we may replace n with j in the upper limit of the sum over k, since the summand vanishes for $k > j$ as well as for $j > n$ due to the binomial factors. However, it is easy to see the following [4, Lemma A.9]:

$$\sum_{k=0}^{j}(-1)^k k\binom{j}{k} = \begin{cases}-1 & \text{if } j=1,\\ 0 & \text{otherwise}.\end{cases} \tag{25}$$

So in Eq. (24), only the summand for $j=1$ remains. Assuming $\rho < 1$, that leads to the result

$$\mathrm{E}[Q_{\text{Segmentation}}] = (1-\rho)^{n-1}. \tag{26}$$

Using Taylor approximation as in Eq. (20), this means that in the limit $n\rho \to \nu > 0$,

$$\mathrm{E}[Q_{\text{Segmentation}}] \to e^{-\nu}; \tag{27}$$

so $Q_{\text{Segmentation}}$ is intensive as well.

IV.D. Vulnerability

In our 1-dimensional model, the importance of a node is either 1 (if removing this node splits the respective network segment in two parts) or 0. As before, we count the connected situation as having 0 segments. We may describe the event j-IMPORTANT (meaning that $I_j = 1$) directly in next-neighbor coordinates as

$$\{\boldsymbol{y} \mid (y_{j-1} < \rho) \wedge (y_j < \rho) \wedge (y_{j-1} + y_j \geq \rho)\}, \tag{28}$$

where the coordinate indices are understood "modulo n," i.e. y_0 is identified with y_n. We assume $n \geq 2$ in the following, so that y_j and y_{j-1} are independent coordinates.

The set in Eq. (28) is alternatively described by the fact that *none* of the conditions $y_{j-1} \geq \rho$, $y_j \geq \rho$, and $y_{j-1} + y_j < \rho$ is fulfilled. Thus, we can apply the well-known inclusion-exclusion formula in order to obtain

$$\begin{aligned} P_{j\text{-IMPORTANT}} &= 1 - P(y_{j-1} \geq \rho) - P(y_j \geq \rho)\\ &- P(y_{j-1} + y_j < \rho) + P(y_j \geq \rho \wedge y_{j-1} \geq \rho)\\ &+ P(y_{j-1} \geq \rho \wedge y_{j-1} + y_j < \rho)\\ &+ P(y_j \geq \rho \wedge y_{j-1} + y_j < \rho)\\ &- P(y_{j-1} \geq \rho \wedge y_j \geq \rho \wedge y_{j-1} + y_j < \rho). \end{aligned} \tag{29}$$

The last three summands of this expression obviously vanish. Further, we can apply Eq. (A2) of the Appendix to calculate

$$P(y_j \geq \rho) = \int d\mu_n^{T-\text{eq}}(\boldsymbol{y})\, \theta(y_j - \rho) = (1-\rho)^{n-1}, \tag{30}$$

and likewise

$$P(y_{j-1} \geq \rho) = (1-\rho)^{n-1}, \tag{31}$$

$$P(y_{j-1} \geq \rho \wedge y_j \geq \rho) = (1-2\rho)^{n-1}; \tag{32}$$

here we assumed $\rho < 1/2$. Finally, Eq. (A3) shows that

$$\begin{aligned} P(y_{j-1} + y_j < \rho) &= \int d\mu_n^{T-\text{eq}}(\boldsymbol{y})\, \theta(\rho - y_{j-1} - y_j)\\ &= 1 - (1-\rho)^{n-2}(1 + (n-2)\rho). \end{aligned} \tag{33}$$

Combining Eqs. (29) – (33), we showed that

$$P_{j\text{-IMPORTANT}} = (n\rho - 1)(1-\rho)^{n-2} + (1-2\rho)^{n-1}. \tag{34}$$

Inserting into Eq. (4), we obtain for $n \geq 2$ and $\rho < 1/2$,

$$\begin{aligned} \mathrm{E}[Q_{\text{Vulnerability}}] &= \frac{1}{n}\sum_{j=1}^{n} P_{j\text{-IMPORTANT}}\\ &= (n\rho - 1)(1-\rho)^{n-2} + (1-2\rho)^{n-1}. \end{aligned} \tag{35}$$

A Taylor approximation (as in the previous sections) leads to the following asymptotic behavior in the limit $n\rho \to \nu > 0$.

$$\mathrm{E}[Q_{\text{Vulnerability}}] \to (\nu - 1)e^{-\nu} + e^{-2\nu}. \tag{36}$$

Thus, vulnerability is an intensive quality measure as well.

IV.E. Reachability

Following its definition in Sec. III, the value of $Q_{\text{Reachability}}$ in our model is

- $(n-1)/n$ in the events 0-DISCONN and 1-DISCONN,
- more generally, $n^{-2}\sum_{i=1}^{k} b_i(b_i - 1)$ in the event k-DISCONN, $k \geq 1$, where b_i are the sizes of the k network segments.

To get a more explicit description of the latter case for $k \geq 2$, we define the events SEGMENT-m-b, where $m \in \{1, \ldots, n\}$, $b \in \{1, \ldots, n-1\}$, which describe that a segment of the network begins exactly at node m, extending "to the right," and has a size of exactly b nodes. (Again, the node coordinates are assumed to be sorted, and the indices are defined modulo n.)

quality measure	expectation value at finite $n \geq 2$, $\rho < 1/2$	expectation value asymptotic ($n \to \infty$)	intensive?
$Q_{\text{Connectedness}}$	$\sum_{j=0}^{[1/\rho]} (-1)^j \binom{n}{j} (1 - j\rho)^{n-1}$	$e^{-(e^{-\eta})}$ where $\eta = n\rho - \ln n$	no
Q_{Coverage}	$1 - (1 - 2\rho)^n$	$1 - e^{-2\nu}$ where $\nu = n\rho$	yes
$Q_{\text{Segmentation}}$	$(1 - \rho)^{n-1}$	$e^{-\nu}$ where $\nu = n\rho$	yes
$Q_{\text{Vulnerability}}$	$(n\rho - 1)(1 - \rho)^{n-2}$ $+(1 - 2\rho)^{n-1}$	$(\nu - 1)e^{-\nu} + e^{-2\nu}$ where $\nu = n\rho$	yes
$Q_{\text{Reachability}}$	see Eqs. (38) and (41)	$2e^{\eta} - (1 + 2e^{\eta})e^{-(e^{-\eta})}$ where $\eta = n\rho - \ln n$	no

Table 1: Overview of the results for quality measures. For the asymptotic formulas, we assumed sequences with $n\rho = \text{const}$ or $n\rho - \ln n = \text{const}$, respectively.

This can be formally expressed by the characteristic function

$$\chi_{\text{SEGMENT-m-b}}(\boldsymbol{y}) = \theta(y_{m-1} - \rho)\, \theta(y_{m+b-1} - \rho) \times \prod_{i=m}^{m+b-2} \theta(\rho - y_i). \quad (37)$$

It is easy to sum over the size of the segments: Since the events SEGMENT-m-b are obviously disjoint from 0-DISCONN and 1-DISCONN, one simply has

$$Q_{\text{Reachability}} = \frac{n-1}{n}(\chi_{\text{0-DISCONN}} + \chi_{\text{1-DISCONN}}) + \sum_{m=1}^{n} \sum_{b=1}^{n-1} \frac{b(b-1)}{n^2} \chi_{\text{SEGMENT-m-b}}. \quad (38)$$

Since the expectation value of the first summand is known from Eq. (15), it only remains to calculate $P_{\text{SEGMENT-m-b}}$ in order to determine $\mathrm{E}[Q_{\text{Reachability}}]$. Using the definition in Eq. (37), and applying Eq. (A1) of the Appendix twice, we see that for $n \geq 2$ and $\rho < 1/2$,

$$\begin{aligned} P_{\text{SEGMENT-m-b}} &= \int d\mu_n^{T-\text{eq}}(\boldsymbol{y})\ \chi_{\text{SEGMENT-m-b}}(\boldsymbol{y}) \\ &= (1 - 2\rho)^{n-1} \int d\mu_n^{T-\text{eq}}(\boldsymbol{y}) \prod_{i=m}^{m+b-2} \theta(\frac{\rho}{1 - 2\rho} - y_i) \\ &= (1 - 2\rho)^{n-1} P(y_m < \rho' \wedge \ldots \wedge y_{m+b-2} < \rho'), \end{aligned} \quad (39)$$

where $\rho' = \rho/(1 - 2\rho)$. For determining the probabilities $P(y_m < \rho' \wedge \ldots)$, we once again use the inclusion-exclusion formula, which yields

$$P(y_m < \rho' \wedge \ldots \wedge y_{m+b-2} < \rho') = \sum_{j=0}^{b-1} (-1)^j \sum\nolimits^* P(y_{m_1} \geq \rho' \wedge \ldots \wedge y_{m_j} \geq \rho'); \quad (40)$$

the sum $\sum^*$ runs over all subsets $\{m_1, \ldots, m_j\} \subset \{m, \ldots, m+b-2\}$. Again, the probability in the sum is known by Eq. (A2) in the Appendix. This leads us to

$$P_{\text{SEGMENT-m-b}} = (1 - 2\rho)^{n-1} \sum_{j=0}^{[1/\rho']} (-1)^j \binom{b-1}{j} (1 - j\rho')^{n-1}. \quad (41)$$

Inserted into the expectation value of Eq. (38), this provides an explicit (but lengthy) expression for $\mathrm{E}[Q_{\text{Reachability}}]$. In the limit $n \to \infty$, however, one can obtain a much simpler form: Setting out from Eq. (41), it can be shown [4, Sec. 4.2.5] that for $n\rho - \ln n \to \eta$,

$$\sum_{m=1}^{n} \sum_{b=1}^{n-1} \frac{b(b-1)}{n^2} P_{\text{SEGMENT-m-b}} \xrightarrow{n \to \infty} 2e^{\eta} - (e^{-\eta} + 2 + 2e^{\eta})e^{-(e^{-\eta})}. \quad (42)$$

Since the limit values of $P_{\text{0-DISCONN}}$ and $P_{\text{1-DISCONN}}$ are known from Eq. (16), we find the asymptotic behavior

$$\mathrm{E}[Q_{\text{Reachability}}] \to 2e^{\eta} - (1 + 2e^{\eta})e^{-(e^{-\eta})} \quad \text{as } n\rho - \ln n \to \eta. \quad (43)$$

Noting that $Q_{\text{Reachability}}$ is monotone in ρ at fixed n, it is clear that $\mathrm{E}[Q_{\text{Reachability}}] \to 0$ as $n\rho \to \nu$; so the reachability is *not* intensive.

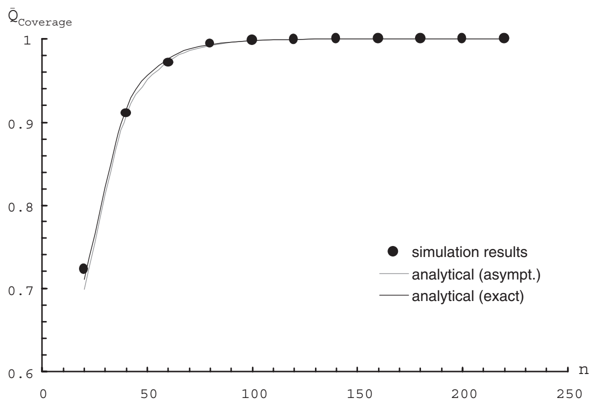


Figure 4: Comparison of results for the coverage

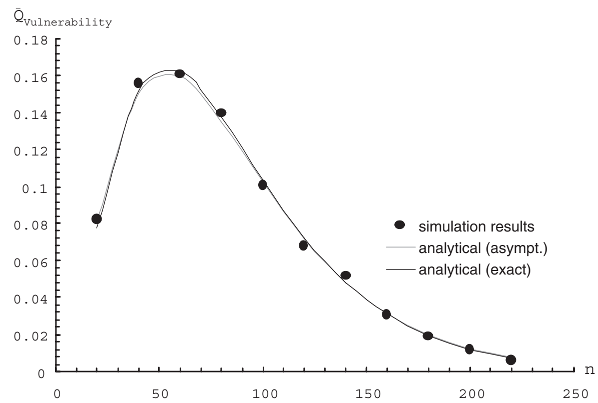


Figure 6: Comparison of results for the vulnerability

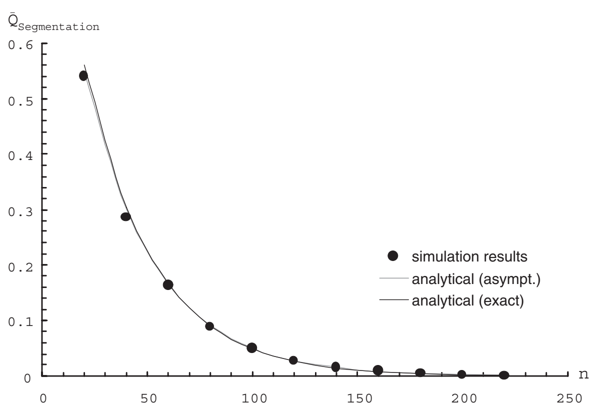


Figure 5: Comparison of results for the segmentation

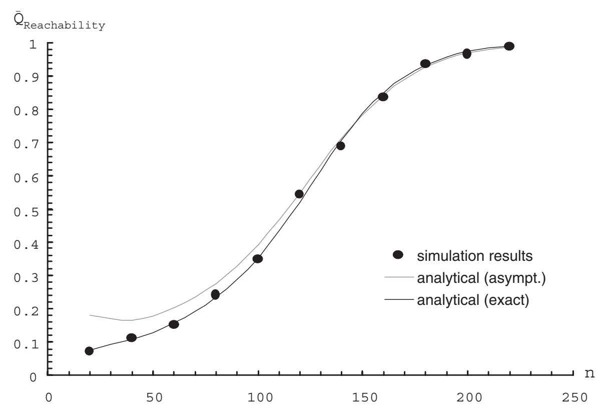


Figure 7: Comparison of results for the reachability

V. Comparison with Simulations

We now wish to test the validity of our analytical results against those of numerical simulations. In contrast to the static situation considered in Sec. IV, we will base the simulations on an explicit motion model for the nodes, and obtain the mean quality levels by averaging over time. One would expect that this setup also leads to a time-independent spatial distribution of nodes in the long run; however, this aspect is not explicitly modeled.

Our first simulation stays very close to the mathematical setting of Sec. IV, except for mobility; i.e., nodes move along a straight line and are able to connect across the boundary. Figures 4 to 7 show the results for $\rho = 0.03$. The figures compare the analytical results (approximation and exact formulas) with simulation data. The simulations confirm the analytical results, as the results match the corresponding values of the exact formulas. In addition, except for smaller n in the reachability chart, the approximation formulas lead to results that are very similar to the exact results.

In a second simulation study, we do not so much aim at a numerical verification within the same mathematical model; rather, we wish to show that our results also remain valid if the modeling context is varied. Therefore, we compare our findings to an apparently different MANET model known in the literature [13].

In contrast to the quite simplistic assumptions of our model, [13] aimed at a more realistic network topology; the simulation is based on the map of a shopping center in Downtown Minneapolis. This shopping center consists of a number of towers which are connected on the first floor via bridges, so-called "Skyways" (Fig. 8); we consider users with wireless devices moving along these paths.

This model is quite similar to ours and mainly makes the same assumptions: Network nodes move independently at random on 1-dimensional paths; the radio range of all nodes is equal with a sharp cutoff at radius r. However, there are a number of important differences.

First, the simulation was based on a 2-dimensional radio propagation, in contrast to our 1-dimensional model; i.e., two nodes are connected whenever their distance is smaller than r on the *plane* rather than

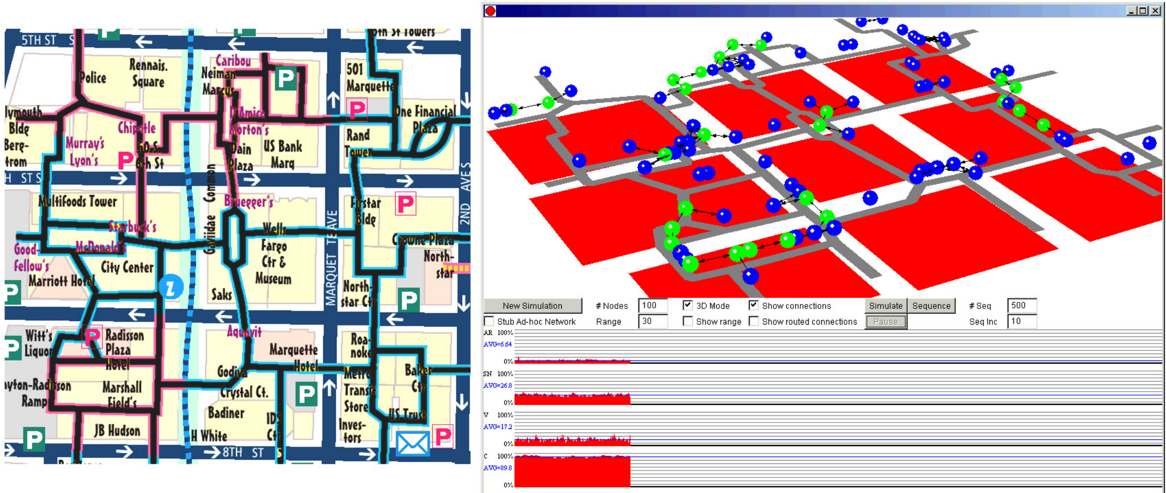


Figure 8: The Minneapolis skyways: original map (left) [8], inside the simulation environment (right)

along the line segments. (No shielding by buildings, walls, etc. between the different paths was taken into account.) In most cases, this is equivalent to our 1-dimensional propagation, since neighboring line segments are usually further than r apart; there are some exceptions however.

Second, the topology of the line segments in [13] is much more complex than in our simplistic model, including both open and closed curves.

In order to compare our results of Sec. IV to those of [13], we first need to fix the appropriate system parameter ρ. The radio range in [13] was chosen as $r = 30$ m (the indoor communication range of IEEE 802.11b Wireless LAN), which we can directly transfer to our situation. However, taking the length ℓ to be the combined length of all paths in the Skyways map (which is 3653 m), it turns out that the simulation results substantially differ from those expected from the analytical formulas. This is due to the 2-dimensional propagation model used in [13], which allows more nodes to connect than our 1-dimensional model predicts.

In order to estimate the order of magnitude of this effect, we introduce the following heuristic procedure. First, whenever two parallel line segments on the map are not further than $r/2$ apart, we count only one of them in determining ℓ. Second, for each point on the map where 3 or more lines meet, we subtract $1r$ from ℓ. (Note that a node which is located near such a point covers an "area" of $3r$ in length in the 2-dimensional propagation model, while it covers only $2r$ when referring to 1-dimensional propagation.) This leads to $\ell = 2463$ m or an effective normalized radio range of $\rho \approx 0.0122$.

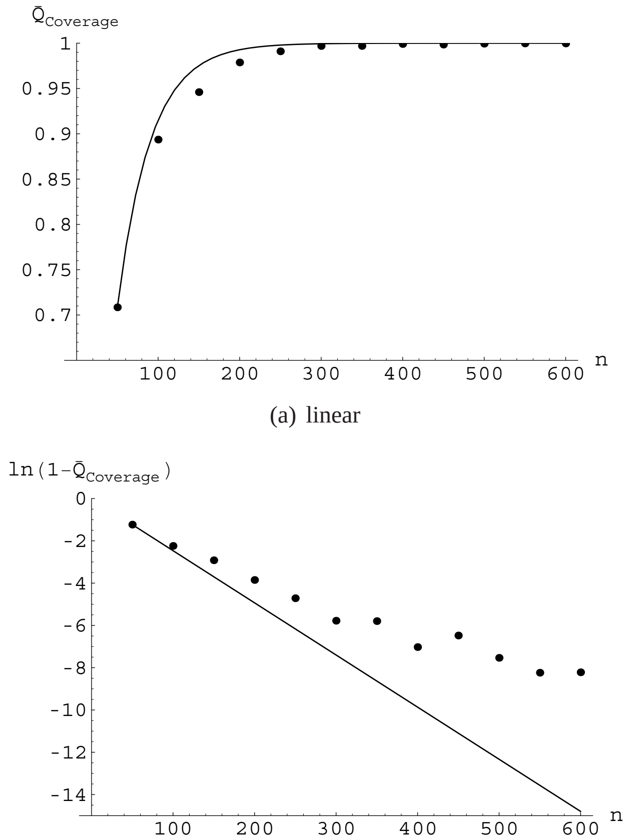
Of course, these heuristic corrections are only very rough and cannot be traced back directly to the statistical description. However, we shall see that with the corrections introduced, we can already get a good match between the results that the two models predict. Alternatively, one might want to determine the effective ρ from a linear fit in Fig. 10(b) below.

After having fixed the system parameters, let us now turn to a direct comparison of the data for area coverage, segmentation, vulnerability, and reachability.[2] For all these quality measures, we compare the numerical results of [13] with our analytical results listed in Table 1, using the exact formulas.

Let us start with area coverage, displayed in Fig. 9. The linear plot shows that both models nearly agree in absolute values for $n = 50$ and $n = 100$, and in the asymptotic behavior as $n \to \infty$ (where both graphs approach 1), while there is some difference at medium values of n. However, the logarithmic plot reveals that our 1-dimensional model systematically differs from the simulation, which shows a much lower area coverage at high n. An explanation for this difference is the boundary effect in the simulation study: as a consequence of the used motion model, peripheral parts of the Skyways, especially the "dead ends" are not as densely covered with nodes as one would expect from the equal distribution. Similar boundary effects are known e.g. from the random waypoint model [16, 3].

The data for segmentation is shown in Fig. 10. It shows a good fit between the models, both on the linear and logarithmic scale. In particular, $\bar{Q}_{\text{Segmentation}}$ decays exponentially with n quite precisely, which is visible in the logarithmic plot; this is exactly the behavior predicted by our simpler model.

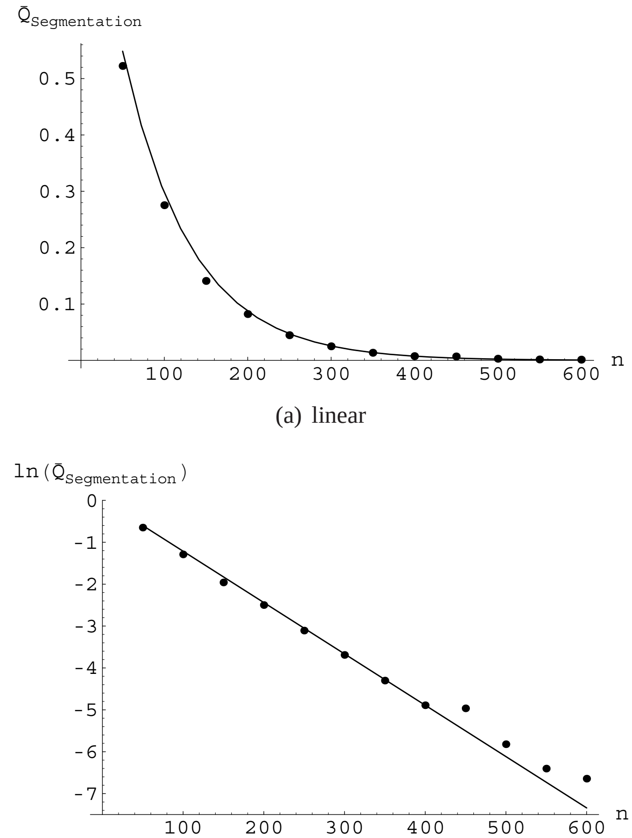
[2] $Q_{\text{Connectedness}}$ was not considered in [13], since connectedness is an exceedingly strict condition for networks of reasonable size.



(a) linear

(b) logarithmic

Figure 9: Analytical and simulation results for area coverage in the Skyways scenario. The numerical data, shown as •, is the same as appeared in [13, Fig. 3a].



(a) linear

(b) logarithmic

Figure 10: Analytical and simulation results for segmentation in the Skyways scenario

Fig. 11 compares the data for $\bar{Q}_{\text{Vulnerability}}$. For this quality measure, we also obtain a good fit between the two models across the range considered for n, except perhaps for the case of very few nodes ($n = 50$).

The last quality measure – reachability – is shown in Fig. 12. While the qualitative behavior agrees between the models also in this case, there are noticeable differences in the absolute value of $\bar{Q}_{\text{Reachability}}$: In the range of medium n, it seems that in the simple 1-dimensional model, approximately 50-100 nodes more are needed to achieve the same reachability as in the simulation. This leads to absolute differences of up to 0.3 in $\bar{Q}_{\text{Reachability}}$ between the models. Taking into account that the average *number* of network segments agrees between the models (cf. Fig. 10), this points to the fact that some particularly large segments occurred in [13] that are not predicted by our 1-dimensional model. The simulation shows that these larger segments are usually located near the "crossings" of different paths; so in this case, our heuristic corrections apparently do not fit as well as for the other quality measures.

It is also worth noting in this context that $Q_{\text{Reachability}}$ is more sensitive to the choice of boundary conditions than the other quality measures discussed. Replacing the periodic boundary conditions of Sec. IV with "disconnected" conditions (i.e., not allowing connections via the boundary), it is easily seen that the resulting differences in $\bar{Q}_{\text{Coverage}}$, $\bar{Q}_{\text{Segmentation}}$, and $\bar{Q}_{\text{Vulnerability}}$ vanish as $n \to \infty$. However, a difference in $\bar{Q}_{\text{Reachability}}$ remains in the limit $n\rho - \ln n \to \eta$, as is suggested by Monte Carlo approximation.

In conclusion, it seems that the numerical results in [13] can be reproduced in our simpler model at least in a qualitative sense, and in large parts also quantitatively. It should be emphasized that this does not amount to a comparison with experiment; we merely compared our results to a different mathematical model, which is partially based on the same simplifying assumptions (e.g. a homogeneous radio range for all nodes). Still, the above comparison may support the claim that the predictions of our explicitly solvable system are stable with respect to some changes in the modeling decisions. Differences with respect to details of the propagation model could be

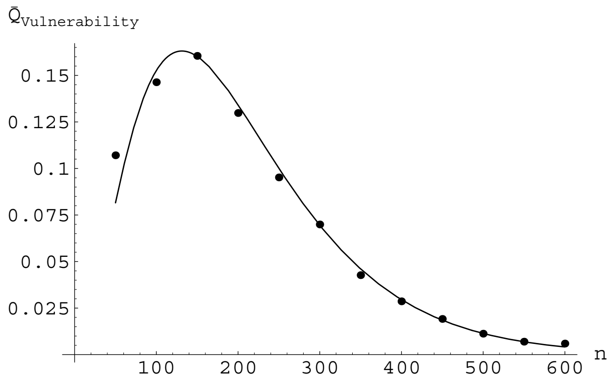


Figure 11: Analytical and simulation results for vulnerability in the Skyways scenario

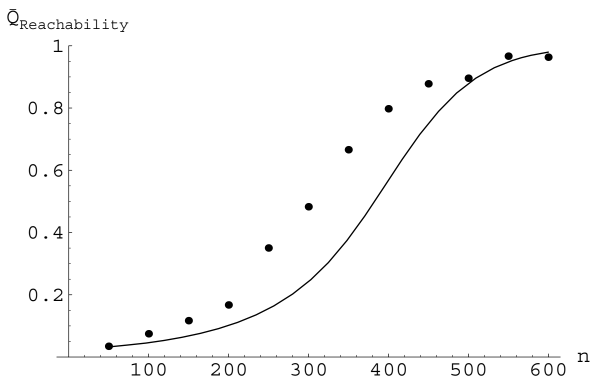


Figure 12: Analytical and simulation results for reachability in the Skyways scenario

compensated by a simple change in the system parameters.

VI. Critical Masses

In this section, we want to reveal a general relation between ρ and n. $\bar{Q}_{\text{Reachability}}$, $\bar{Q}_{\text{Coverage}}$ and $\bar{Q}_{\text{Segmentation}}$ are monotone in n at fixed ρ, and converge to 1, 1, and 0, respectively, as $n \to \infty$. $\bar{Q}_{\text{Vulnerability}}$ is not monotone, but also converges for large n. Thus, it is reasonable to look for an n for which the MANET attained certain quality values.

As a MANET should not only have a single quality property, we combine all four measures in a single formula. The *critical mass* for a specific ρ is the number of nodes required to get appropriate results for all four measures. The level of quality is expressed by a number $q \in [0, 1]$. Higher values for q specify higher

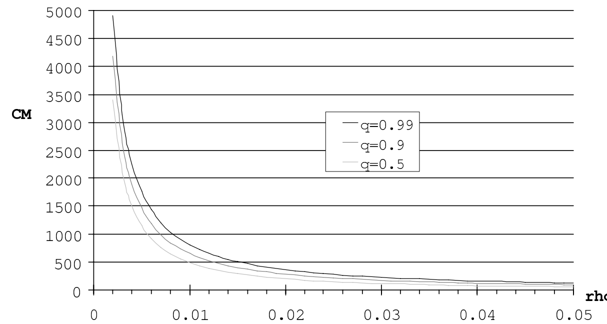


Figure 13: Analytical results for the critical mass

quality demands. We define the critical mass CM_q as

$$\begin{aligned} &\mathrm{CM}_q : \mathbb{R} \to \mathbb{N}, \\ &\quad \rho \mapsto \min\{n \mid \min\{\bar{Q}_{\text{Coverage}}, \bar{Q}_{\text{Reachability}}, \\ &\quad 1 - \bar{Q}_{\text{Vulnerability}}, 1 - \bar{Q}_{\text{Segmentation}}\} \geq q\}. \end{aligned} \tag{44}$$

Figure 13 presents the function CM_q for three values of q.

CM_q provides a general guideline to decide whether a considered MANET will provide sufficient quality or not. One observation is that a relatively high number of nodes is required to get a useful MANET. As an example, consider $\ell = 2000$ m and $r = 10$ m (Bluetooth), thus $\rho = 0.005$. For a quality level of $q = 0.9$, we need $\mathrm{CM}_q = 1463$ nodes to get a reasonable MANET.

VII. The Multi-dimensional Case

Our results in the previous sections were limited to 1-dimensional MANETs, or (in the Skyways example) to special 2-dimensional systems which become effectively 1-dimensional, since the network nodes travel along fixed paths. The present authors are convinced, however, that the main concepts of this paper, in particular the classification of intensive vs. non-intensive quality measures, are applicable to general 2- or 3-dimensional MANETs as well.

To illustrate this, let us note that the analytical results for area coverage can be generalized to higher dimensions: The methods of Sec. IV.B carry over quite directly to an s-dimensional MANET, where nodes are equally distributed over $[0, \ell]^s$. This leads us to the result

$$\mathrm{E}[Q_{\text{Coverage}}] = 1 - (1 - c_s \rho^s)^n, \tag{45}$$

where c_s is the volume of the unit sphere in s dimensions ($c_1 = 2$, $c_2 = \pi$, $c_3 = \frac{4}{3}\pi$). In the limit

$n\rho^s \to \nu$, we obtain

$$\mathrm{E}[Q_{\text{Coverage}}] \to 1 - e^{-c_s \nu}, \tag{46}$$

meaning that Q_{Coverage} is still an intensive quality measure in the sense of Sec. III.C.

Regarding our results for other quality measures, however, it does not seem straightforward to generalize the techniques used; in particular, the formalism of next-neighbor coordinates is not available beyond $s = 1$.

Still, some qualitative results can be deduced from the literature: For a 2-dimensional MANET (in a slightly different setup compared to ours), Bettstetter has shown [2] that the probability that no node is isolated from the network is given by

$$P(\text{no node isolated}) = (1 - e^{-n\pi\rho^2})^n \tag{47}$$

when expressed in our notation.[3] In the limit $n \to \infty$, $n\rho^2 \to \nu$, this probability obviously vanishes. Since a network with an isolated node is certainly not connected, we obtain in particular that $\bar{Q}_{\text{Connectedness}} \to 0$ in that limit. Thus, $Q_{\text{Connectedness}}$ is non-intensive in 2-dimensional MANETs as well. Qualitatively, this is also in line with results by Xue and Kumar [15].

VIII. Conclusions and Future Work

In this paper, we analyzed a number of different quality measures for MANETs. In general, quality measures can be classified into intensive measures (with good scaling properties) and non-intensive ones (which possibly lead to scalability problems). We were able to obtain explicit results for all of the measures in a simple 1-dimensional model. These results agree with simulation data. Our analytic methods should be applicable also to other quality measures in the same model, as long as they can reasonably be expressed in terms of next-neighbor distances.

Our results can serve both as a qualitative and quantitative guideline for the design of 1-dimensional MANET systems, as well as for sensor networks [1]. Due to our explicit results for the expectation value of quality measures, it is easy to choose the radio range or node density in a MANET such that it reaches the desired quality level. In particular, this applies to the asymptotic formulas; they are certainly simple enough to even allow computation on the mobile devices themselves.

[3] Note that in the notation used in [2], the symbol ρ does not represent the normalized radio range, but rather the density of the Poisson process used.

Certainly, we merely treated a small part of the problems and obstacles that may limit the quality and scalability of MANETs, and neglected e.g. questions of routing or end-to-end throughput. Thus, our results should be regarded as an *upper bound* to MANET quality, in the sense that additional problems might be faced on higher layers.

Also, we only derived our analytical results for static deployments of network nodes. While for the quality measures we considered, we are in good agreement with simulations that include mobility (cf. Sec. V), our framework does currently not allow to discuss aspects that are directly linked to the time evolution of the system, such as the question: "What is the probability that the network is connected for a time frame of length t_0?"

In order to answer such questions, we need to make specific assumptions on the motion of nodes. It would in principle be possible to incorporate one of the common explicit mobility models [5] into our context. From a more general point of view, however, these explicit models seem to be rather ad hoc and are not really motivated by properties of the real network system. They also include technicalities that might complicate an analytical approach more than necessary. Therefore, it might be desirable to consider a model with more natural assumptions, or to explore and compare several such modeling alternatives.

Appendix

In our analysis, we need to calculate certain integrals over the manifold T_n, as defined in Eq. (12). This appendix compiles some useful formulas to that end. One can obtain these results by elementary integration; the reader is referred to [4, Appendix A.1] for a more detailed exposition.

In the following, let $n \in \mathbb{N}$ and $\lambda \in (0,1)$. First, we present a scaling argument related to the integration of theta functions: For any $j \in \{1, \ldots, n\}$ and any integrable function $f : T_n \to \mathbb{R}$, one has

$$\int d\mu_n^{T-\mathrm{eq}}(\boldsymbol{y})\; \theta(y_j - \lambda) f(\boldsymbol{y}) = (1-\lambda)^{n-1} \int d\mu_n^{T-\mathrm{eq}}(\boldsymbol{y})\; f((1-\lambda)\boldsymbol{y} + \lambda \boldsymbol{e}_{(j)}), \tag{A1}$$

where $\boldsymbol{e}_{(j)} \in \mathbb{R}^n$ is the j-th standard unit vector. By repeated use of this relation, one can obtain for any m-element subset $\{j_1, \ldots, j_m\} \subset \{1, \ldots, n\}$ that

$$\int d\mu_n^{T-\mathrm{eq}}(\boldsymbol{y}) \prod_{i=1}^{m} \theta(y_{j_i} - \lambda) = (1 - m\lambda)^{n-1}, \tag{A2}$$

provided that $m\lambda \leq 1$; for $m\lambda > 1$, the left-hand side vanishes.

Further, let $j, k \in \{1, \ldots, n\}$ with $j \neq k$. Then it holds that

$$\int d\mu_n^{T-\mathrm{eq}}(\boldsymbol{y})\, \theta(\lambda - y_j - y_k) = 1 - (1-\lambda)^{n-2}(1 + (n-2)\lambda). \quad \text{(A3)}$$

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